

# Gauge/Bethe correspondence on $S^1 \times \Sigma_h$ and index over moduli space

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## Abstract

We introduce two-types of topologically twisted Chern-Simons-matter theories on the direct product of circle and genus- $h$  Riemann surface ( $S^1 \times \Sigma_h$ ). The partition functions of first model agrees with the partition functions of a generalizations of  $G/G$  gauged WZW model. We also find that correlation functions of Wilson loops in first type Chern-Simons-matter theory coincide with correlation functions of  $G$  elements in the generalization of  $G/G$  gauged WZW model. The partition function of this model also has nice interpretations as norms of eigen states of Hamiltonian in the quantum integrable model (q-boson hopping model) and also as a geometric index over a particular moduli space. In the second-type Chern-Simons-matter theory, the partition function is related to integration over moduli space of Hitchin equation on Riemann surface.

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## 1 Introduction

$G$  Wess-Zumino-Witten (WZW) model is a non-linear sigma model whose target space is given by a Lie group  $G$ . In two dimensions,  $G$  WZW model has been studied extensively in the context of conformal field theory.  $G/G$  gauged WZW model we concern is obtained by gauging Lie group  $G$  in  $G$  WZW model.  $G/G$  gauged WZW model also possesses many interesting structures. For examples:

1. The partition function  $\mathcal{Z}_{\text{GWZW}}(\Sigma_h)$  of  $G/G$  gauged WZW model on genus  $h$  Riemann surface  $\Sigma_h$  is expressed by modular  $S$ -matrices [1, 2] as

$$\mathcal{Z}_{\text{GWZW}}(\Sigma_h) = \sum_{\mu} \frac{1}{S_{0\mu}^{2h-2}} \quad (1.1)$$

2.  $G/G$  gauged WZW model is a two dimensional topological quantum field theory [3].

3.  $G/G$  gauged WZW model possesses a hidden quantum integrable structure [4, 5]

4. The partition function  $\mathcal{Z}_{\text{GWZW}}(\Sigma_h)$  is positive integer which is given by an index over the moduli space of  $G$  flat connections on Riemann surface  $\Sigma_h$  [6].

5.  $\mathcal{Z}_{\text{GWZW}}(\Sigma_h)$  is identical to the partition function  $\mathcal{Z}_{\text{CS}}(S^1 \times \Sigma_h)$  of  $G$  pure Chern-

Simons theory on  $S^1 \times \Sigma_h$ :

$$\mathcal{Z}_{\text{GWZW}}(\Sigma_h) = \mathcal{Z}_{\text{CS}}(S^1 \times \Sigma_h). \quad (1.2)$$

In our previous work [7], we constructed a one parameter generalization of  $G/G$  gauged WZW model ( $G/G$  gauged WZW-matter model) and evaluated its partition function and correlation functions. We showed that  $G/G$  gauged WZW-matter model possesses following similar properties to  $G/G$  gauged WZW model:

1'. The partition function  $\mathcal{Z}_{\text{GWZWM}}(\Sigma_h)$  of  $G/G$  gauged WZW model on genus  $h$  Riemann surface  $\Sigma_h$  is expressed by one-parameter deformation of modular  $S$ -matrices as

$$\mathcal{Z}_{\text{GWZWM}}(\Sigma_h) = \sum_{\mu} \frac{1}{S_{0\mu}^{2h-2}(t)}. \quad (1.3)$$

2'.  $G/G$  gauged WZW-matter model is a two dimensional topological quantum field theory.

3'.  $G/G$  WZW-matter model model possesses a hidden quantum integrable structure.

From an analogy of gauged WZW model, we also suggested following conjectures:

4'. The partition function  $\mathcal{Z}_{\text{GWZWM}}(\Sigma_h)$  is related to a partition function of topological twisted Chern-Simons-matter theory on  $S^1 \times \Sigma_h$  constructed in [8].

5'. The partition function of  $G/G$  WZW-matter model is related to a geometric index over moduli space [9].

One purpose of this paper is to reveal the relation between  $G/G$  gauged WZW-matter model on  $\Sigma_h$  and twisted Chern-Simons-matter theory on  $S^1 \times \Sigma_h$ . The second motivation of this paper is to relate the twisted Chern-Simons-matter theories to integration over moduli spaces defined particular differential equations. In the case of pure Chern-Simons theory, it is not difficult to see the path integral localize to the moduli space of flat connections. But, in the previous work, it was not clear how the path integral of  $G/G$  gauged WZW-matter model localized to moduli spaces. The third motivation is Gauge/Bethe correspondence discovered in [11, 12]. So far, it has not been well studied that what types of objects in the quantum integrable model correspond to the partition

functions or correlation functions in the gauge theory side. In [7], we revealed that the partition function (correlation functions) of  $G/G$  gauged WZW-matter is related to the norms of eigen states of Hamiltonian ( the correlation functions of conserved charges ) in the quantum integrable model, respectively. In this case, the corresponding quantum integrable model is  $q$ -boson hopping model [13]. From the above conjectures, we expect that the partition function and correlation functions in the twisted Chern-Simons-matter theory are also related to norms of eigen states and correlation functions in the quantum integrable model.

This paper is organized as follows. In section 2, we introduce two types of twisted Chern-Simon-matter theories. Both of two models can be embedded into  $S^1$ -uplift of A-twisted Chern-Simons-matter theories. The difference between two models is R-charge assignment before twisting. In subsection 2.1, we introduce a twisted Chern-Simons-matter theory on  $S^1 \times \Sigma_h$  which is related to  $G/G$  gauged WZW-matter model on  $\Sigma_h$ . We evaluate the partition function and show the equivalence between the twisted Chern-Simons-matter theory and  $G/G$  gauged WZW-matter model. This means that Chern-Simons-matter theory also possesses quantum integrable structure and TQFT structure. In subsection 2.2, we introduce another twisted Chern-Simons-matter theory. In subsection 2.3, we generalize the correspondence between the partition functions to correlation functions. Wilson loop correlation functions in the first type twisted Chern-Simons-matter theory correspond to correlation functions of  $G$ -elements in the  $G/G$  gauged WZW-matter models. In section 3, we study the interpretation of the partition functions of two Chern-Simons-matter theories in terms of moduli space. The last section is devoted to summary.

## 2 Twisted Chern-Simons matter theory on $S^1 \times \Sigma_h$

In this section we evaluate twisted Chern-Simons-Matter theories on the direct product of circle and genus- $h$  Riemann surface  $S^1 \times \Sigma_h$ . The model consists of Chern-Simons term and the topologically twisted matter sectors. The twist we introduce is  $S^1$ -uplift of A-twist in two dimensions [8]. We take local coordinate of  $S^1 \times \Sigma_h$  as  $(x^0, x^1, x^2) = (t, z, \bar{z})$  and take circumference of  $S^1$  as  $2\pi$ .

First we consider Chern-Simons term. In order to introduce matter multiplets in the BRST exact manner. We introduce adjoint 1-form valued fermion  $\lambda_i$ , ( $i = 1, 2$ ) and scalar Grassmann odd symmetry  $Q$  (BRST symmetry). When we do not couple Chern-Simons theory with twisted matters,  $\lambda_i$  are auxiliary fields and do not affect theory explicitly.

The  $Q$ -transformation is defined by

$$\begin{aligned} QA_j &= \lambda_j, \quad Q\lambda_j = F_{tj}, \quad (j = 1, 2), \\ QA_t &= 0. \end{aligned} \tag{2.1}$$

The Chern-Simons action is defined by

$$\begin{aligned} S_{\text{CS}}[A] &= \frac{ik}{4\pi} \int_{S^1 \times \Sigma_h} \text{Tr} \left[ A \wedge dA + \frac{2i}{3} A \wedge A \wedge A \right] \\ &= \frac{ik}{4\pi} \int_{S^1 \times \Sigma_h} d^3x \epsilon^{\mu\nu\rho} \text{Tr} \left[ A_\mu \partial_\nu A_\rho + \frac{2i}{3} A_\mu A_\nu A_\rho \right]. \end{aligned} \tag{2.2}$$

The  $Q$ -closed completion of Chern-Simons action is given by

$$S_{\text{coh}}[A, \lambda] = S_{\text{CS}}[A] + \frac{ik}{4\pi} \int_{S^1 \times \Sigma_h} d^3x \text{Tr} [\epsilon^{ij} \lambda_i \lambda_j]. \tag{2.3}$$

The bosonic part of  $S_{\text{coh}}$  is written as

$$S_{\text{CS}}[A] = \frac{ik}{4\pi} \int_{S^1 \times \Sigma_h} \epsilon^{ij} \text{Tr} [A_t F_{ij} - A_i \partial_t A_j], \tag{2.4}$$

Integrating out a gauge field  $A_t$  in the pure Chern-Simons action imposes a delta functional constraint  $\delta(F_{12})$  and the partition function is interpreted as partition function of quantum mechanical system whose target space is the moduli space of the flat connection condition  $F_{12} = 0$  on the Riemann surface. Then the quantization of Chern-Simons theory on  $S^1 \times \Sigma_h$  is the quantization problem of moduli space of flat connection on Riemann surface  $\Sigma_h$ . We will see the similar interpretation is possible for twisted Chern-Simons-Matter theories in section 3.

Next, we couple Chern-Simons theory to matters fields. We are interested in one-parameter deformation of Chern-Simons theory which are equivalent to one-parameter deformation of  $G/G$  gauged WZW model constructed in [7] and [10].

## 2.1 The model 1

First we consider a deformation of Chern-Simons theory which is equivalent to  $G/G$  gauged WZW-matter model introduced in [7]. The matter fields we introduce are following:

- The Grassmann even matter:  $(\phi, \bar{\phi}, Y_{\bar{z}}, Y_z)$
- The Grassmann odd matter  $(\psi, \bar{\psi}, \chi_{\bar{z}}, \chi_z)$

$\phi$  and  $\bar{\phi}$  are complex scalar,  $Y_z, \chi_z$  are  $(1, 0)$ -form and  $Y_{\bar{z}}, \chi_{\bar{z}}$  are  $(0, 1)$ -form along Riemann surface, respectively. All the matter fields belong to complexified adjoint representation of the Lie algebra  $\mathfrak{g}$ . The BRST transformation is defined by

$$\begin{aligned} Q\phi &= \psi, & Q\psi &= D_t\phi + m\phi, \\ QY_{\bar{z}} &= D_t\chi_{\bar{z}} - m\chi_{\bar{z}}, & Q\chi_{\bar{z}} &= Y_{\bar{z}}, \\ Q\bar{\phi} &= \bar{\psi}, & Q\bar{\psi} &= D_t\bar{\phi} - m\bar{\phi}, \\ QY_z &= D_t\chi_z + m\chi_z, & Q\chi_z &= Y_z. \end{aligned} \quad (2.5)$$

$D_t\phi := \partial_t\phi + i[A_t, \phi]$  and so on. Here  $m$  is a equivariant mass parameter (real mass) which regularizes the flat direction. When we perform the dimensional reduction along the  $S^1$ -direction. Then (2.5) reduces to the BRST transformation of A-twisted chiral multiplet on the Riemann surface. We introduce the action for matter fields as  $Q$ -exact matter:

$$\begin{aligned} S_{\text{mat}1} &= \int_{S^1 \times \Sigma_h} Q\text{Tr} \left[ \bar{\phi}\psi + g^{\bar{z}z}(Y_{\bar{z}}\chi_z) \right] \\ &= \int_{S^1 \times \Sigma_h} \text{Tr} \left[ \bar{\phi}(D_t\phi + m\phi) + \bar{\psi}\psi + g^{z\bar{z}}Y_zY_{\bar{z}} - g^{\bar{z}z}\chi_{\bar{z}}(D_t\chi_z + m\chi_z) \right]. \end{aligned} \quad (2.6)$$

The action is again invariant under the  $Q$ -transformation. The partition function of twisted Chern-Simons-Matter theory is defined by

$$\mathcal{Z}_{\text{CSM1}}(S^1 \times \Sigma_h) = \frac{1}{\text{Vol}(\mathcal{G})} \int \mathcal{D}^3 A_\mu \mathcal{D}^2 \lambda_z \mathcal{D}^2 \phi \mathcal{D}^2 \psi \mathcal{D}^2 Y_i \mathcal{D}^2 \chi_i e^{S_{\text{coh}} - S_{\text{mat}}}. \quad (2.7)$$

Here  $\mathcal{G}$  is the gauge transformation group of  $G$  and  $\text{Vol}(\mathcal{G})$  is it volume. We evaluate the partition function. We take the Cartan-Weyl basis  $(H_a, E_\alpha)$  of the Lie algebra  $\mathfrak{g}$  of gauge group  $G$ . The commutation relations in the Cartan-Weyl basis are

$$[H^a, H^b] = 0, \quad [H^a, E^\alpha] = \alpha^a E^\alpha \quad (2.8)$$

Trace is defined as

$$\text{Tr}(H^a H^b) = \delta_{ab}, \quad \text{Tr}(E^\alpha E^\beta) = \frac{2}{|\alpha|^2} \delta^{\alpha+\beta, 0}. \quad (2.9)$$

Let  $\Psi$  is a generic Lie algebra valued field. Then,  $\Psi$  is expanded by the Cartan-Weyl basis as follows.

$$\Psi = \sum_a \Psi^a H^a + \sum_{\alpha \neq 0} \Psi^\alpha E^\alpha. \quad (2.10)$$

From now on, we assume gauge group is  $U(N)$ . We take following gauge [16]

$$\partial_t A_t^a = 0 \quad \text{and} \quad A_t^\alpha = 0. \quad (2.11)$$

The first condition in (2.11) requires the gauge field along time direction depends only on the coordinates on Riemann surface. Then ghost action associated to the above gauge fixing condition is given by

$$S_{\text{gh}} = \int_{S^1 \times \Sigma_h} \text{Tr}(\bar{c} \partial_t c + i \bar{c} [A_t^a H^a, c]). \quad (2.12)$$

Next we expand the three dimensional fields by Kaluza-Klein (KK) modes along the  $S^1$ -direction as

$$\Psi(t, z, \bar{z}) = \sum_{n \in \mathbb{Z}} \Psi_n(z, \bar{z}) e^{int}. \quad (2.13)$$

By integrating out  $(c, \bar{c})$ , the one-loop determinant of ghost action is formally written as

$$\begin{aligned} Z_{\text{ghost}}^{1\text{-loop}} &= \prod_{n \in \mathbb{Z}} \prod_{\alpha \neq 0} \text{Det}_{\Omega(0,0)}(in + i\alpha(A_t)) \\ &= \prod_{\alpha \neq 0} \text{Det}_{\Omega(0,0)}(1 - e^{2\pi i \alpha(A_t)}) \end{aligned} \quad (2.14)$$

Here  $n \in \mathbb{Z}$  expresses the KK modes along the  $S^1$ -direction and  $\text{Det}_{\Omega(n,m)}$  is functional determinant associated to the section of  $(n, m)$ -form on the Riemann surface. We can also integrate out the gauge field along Riemann surface  $A_t^\alpha$ , then we obtain the functional determinant:

$$\begin{aligned} Z_{A_i}^{1\text{-loop}} &= \prod_{n \in \mathbb{Z}} \prod_{\alpha \neq 0} \text{Det}_{\Omega(1,0)}(ni + i\alpha(A_t))^{-1} \\ &= \prod_{\alpha > 0} \text{Det}_{\Omega(1,0)}(1 - e^{2\pi i \alpha(A_t)})^{-1} \text{Det}_{\Omega(0,1)}(1 - e^{-2\pi i \alpha(A_t)})^{-1} \end{aligned} \quad (2.15)$$

Each functional determinant (2.15) and (2.14) is infinite dimensional and not well-defined, but the ratio between these determinants are evaluated by using heat kernel regularization as

$$\begin{aligned} &\prod_{\alpha > 0} \frac{\text{Det}_{\Omega(0,0)}(1 - e^{2\pi i \alpha(A_t)})}{\text{Det}_{\Omega(1,0)}(1 - e^{2\pi i \alpha(A_t)})} \frac{\text{Det}_{\Omega(0,0)}(1 - e^{-2\pi i \alpha(A_t)})}{\text{Det}_{\Omega(0,1)}(1 - e^{-2\pi i \alpha(A_t)})} \\ &= \prod_{\alpha \neq 0} \exp \left\{ \frac{1}{8\pi} \int_{\Sigma_h} R \log(1 - e^{2\pi i \alpha(A_t)}) + \frac{1}{2\pi} \int_{\Sigma_h} \alpha_a F^a \log(1 - e^{2\pi i \alpha(A_t)}) \right\}. \end{aligned} \quad (2.16)$$

Here  $R$  is the scalar curvature associated to the metric on Riemann surface and  $F_a, (a = 1, \dots, \text{rank}(\mathfrak{g}))$  are field strengths of the Cartan part of gauge field.

Next we evaluate the functional determinant from the matter fields. By integrating out  $(\bar{\phi}, \phi)$  and  $(\chi_z, \chi_{\bar{z}})$ , the ratio of functional determinant are again evaluated by heat kernel regularization as

$$\begin{aligned} \frac{\text{Det}_\chi(\partial_t + i\alpha(A_t) - m)}{\text{Det}_\phi(\partial_t + i\alpha(A_t) - m)} &= (1 - e^{-2\pi m})^{N(h-1)} \\ &\times \prod_{\alpha \neq 0} \exp \left\{ -\frac{1}{8\pi} \int_{\Sigma_h} R \log(1 - e^{2\pi(i\alpha(A_t) - m)}) - \frac{1}{2\pi} \int_{\Sigma_h} \alpha_a F^a \log(1 - e^{2\pi(i\alpha(A_t) - m)}) \right\}. \end{aligned} \quad (2.17)$$

From (2.16) and (2.17) The partition function (2.7) can be written as

$$\begin{aligned} \mathcal{Z}_{\text{CSM1}}(S^1 \times \Sigma_h) &= (1 - e^{-2\pi m})^{N(h-1)} \int \mathcal{D}A^{a3} \mathcal{D}\lambda^{a2} \prod_{a \neq b} \exp \left\{ \frac{1}{8\pi} \int_{\Sigma_h} R \log \left( \frac{1 - e^{(2\pi i(A_t^a - A_t^b))}}{1 - e^{2\pi i(A_t^a - A_t^b + im)}} \right) \right\} \\ &\times \exp \left[ \int_{\Sigma_h} \sum_{a=1}^N i\beta(A_t^a) F^a - \int_{S^1 \times \Sigma_h} \frac{ik}{4\pi} \epsilon^{ij} \lambda_i^a \lambda_j^a \right] \end{aligned} \quad (2.18)$$

with

$$\beta^a(x) := kx^a - \sum_{\substack{b=1 \\ b \neq a}} \frac{i}{2\pi} \log \frac{e^{2\pi i x^a} - e^{-2\pi m} e^{2\pi i x^b}}{e^{-2\pi m} e^{2\pi i x^a} - e^{2\pi i x^b}} \quad (2.19)$$

In [7], we conjectured the existence of twisted Chern-Simons-Matter theory whose partition function is identical to  $G/G$  gauged WZW-matter model. Especially, we expected that the equivariant coupling constant which regularize the flat direction in the gauged WZW-matter model  $t$  (not time coordinate) is related to mass parameter in the corresponding Chern-Simons-Matter theory. In fact (2.18) is precisely same as the Abelianized effective action of  $G/G$  gauged WZW-matter model in [7] with the identification  $e^{-2\pi m} = t$ . Therefore we find that partition function of the Chern-Simons-matter (2.32) agrees with the the partition function of  $U(N)/U(N)$  gauged WZW-matter model  $\mathcal{Z}_{\text{GWZWN}}$  on genus  $h$  Riemann surface:

$$\mathcal{Z}_{\text{CSM1}}(S^1 \times \Sigma_h) = \mathcal{Z}_{\text{GWZWN}}(\Sigma_h). \quad (2.20)$$

The evaluation of the path integral of  $\mathcal{D}A_a^3 \mathcal{D}\lambda_a^2$  is same as that of partition function of  $G/G$  gauged WZW-matter model. So we only briefly mention on the derivation. We decompose the Abelianized field strength to harmonic part  $F_{(0)}^a$  and a exterior derivative



of some one-form part  $\tilde{A}^a$  as  $F^a = F_{(0)}^a + d\tilde{A}^a$ . Here  $F^a = F_{(0)}^a$  satisfies Dirac quantization condition

$$k^a = \frac{1}{2\pi} \int_{\Sigma_h} F_{(0)}^a, \quad k^a \in \mathbb{Z}. \quad (2.21)$$

The path integral of  $\tilde{A}^a$  with the partial integration  $d\tilde{A}^a(\cdots) = \tilde{A}^a d(\cdots)$  imposes  $A_t^a$  to constant configuration  $\varphi^a$ . By using Poisson resummation formula, the summation over magnetic charges  $k_a$ , imposes the field configuration  $\varphi^a$  to satisfy the solutions

$$\beta_1^a(\varphi) = n^a, \quad n^a \in \mathbb{Z}. \quad (2.22)$$

This equation (2.22) is precisely same as the logarithmic form of Bethe Ansatz equation of  $q$ -boson hopping model. Taking account into BRST completion of gaugino harmonic modes, the partition function is evaluated as

$$\begin{aligned} \mathcal{Z}_{\text{CSM1}}(S^1 \times \Sigma_h) &= (1 - e^{-2\pi m})^{N(h-1)} \\ &\times \sum_{\{\varphi^a\}_{a=1}^N \in \mathcal{B}_1} \left( \left| \det_{c,d} \left[ \frac{\partial \beta_1^c(\varphi)}{\partial \varphi^d} \right] \right| \prod_{a \neq b} \frac{e^{2\pi i \varphi_a} - e^{-2\pi m} e^{2\pi i \varphi_b}}{e^{2\pi i \varphi_a} - e^{2\pi i \varphi_b}} \right)^{h-1} \end{aligned} \quad (2.23)$$

Here  $\mathcal{B}_1$  is the set of the solutions of the Bethe Ansatz equations (2.22). A overall  $\varphi^a$  independent ambiguity exists, but such a ambiguity does not affect the correlation functions normalized by the partition function. In the the  $G/G$  gauged WZW-matter model side,  $\varphi^a$  comes from the constant configuration of diagonalized  $G$ -element  $g = e^{2\pi i \sum_a \varphi^a H^a} : \Sigma_h \rightarrow U(1)^N \subset U(N)$ .

In the dictionary of Gauge/Bethe correspondence [12], the saddle point equation of effective twisted super potential of three dimensional  $\mathcal{N} = 2$  supersymmetric theory on  $S^1 \times \mathbb{R}^2$  gives Bethe Ansatz equation in the corresponding quantum integrable system. As pointed out in [14], The saddle point equation of the twisted superpotential in  $\mathcal{N} = 2$  supersymmetric Chern-Simons-matter theory with an adjoint chiral multiplet give the Bethe Ansatz equation of  $q$ -boson model. Actually, before the topologically twisting, the matter content is same.

Next we mention about the properties of twisted Chern-Simons-matter theory.

- Gauge/Bethe correspondence

In the previous paper [7], we showed that the partition function of  $U(N)/U(N)$  gauged WZW-matter model  $\mathcal{Z}_{\text{GWZWM}}$  is expressed as norms of eigen states of Hamil-

tonian in the  $q$ -boson model:

$$\begin{aligned}\mathcal{Z}_{\text{GWZWM}}(\Sigma_h) &= \sum_{\{\varphi^a\}_{a=1}^N \in \mathcal{B}_1} \langle \psi_N(\{e^{2\pi i \varphi^a}\}, t) | \psi_N(\{e^{2\pi i \varphi^a}\}, t) \rangle^{h-1} \\ &= \sum_{\{\varphi^a\}_{a=1}^N \in \mathcal{B}_1} ||\psi_N(\{e^{2\pi i \varphi^a}\}, t)||^{2h-2}.\end{aligned}\tag{2.24}$$

Here  $|\psi_N(\{e^{2\pi i \varphi^a}\}, t)\rangle$  are the eigen vectors of Hamiltonian of  $q$ -boson model in the  $N$ -particle sector and  $\langle \psi_N(\{e^{2\pi i \varphi^a}\}, t)|$  is the dual vector of  $|\psi_N(\{e^{2\pi i \varphi^a}\}, t)\rangle$ . In the  $q$ -boson model side,  $t$  corresponds to the  $q$ -deformation parameter. The limit  $t \rightarrow 0$  corresponds to the strong coupling limit. In this limit  $q$ -boson model reduces to phase model. In the gauged WZW-matter (Chern-Simons-matter model) side, matter decouples from the WZW model (pure Chern-Simons theory) in the limit  $t \rightarrow 0, (m \rightarrow \infty)$ . The gauge/Bethe correspondence between  $G/G$  gauged WZW model (pure Chern-Simons theory) and phase was studied in [5]. Again the partition function of  $G/G$  gauged WZW model on  $\Sigma_h$  is expressed by the norms of eigen states in the phase model.

- Deformed Verlinde formula and TQFT structure

In [7] we also showed that the partition function of gauged WZW-matter model is expressed by one-parameter deformation of modular  $S$ -matrices  $S_{\mu\nu}(t)$ ,  $(\mu, \nu \in \mathcal{A}_{N,k}^+)$  introduced by Korff [15] and that the partition function  $\mathcal{Z}_{\text{GWZWM}}$  or equivalently  $\mathcal{Z}_{\text{CSM1}}$  can be constructed from axiom of two dimensional topological quantum field theory (TQFT) [17, 18]. Here  $\mathcal{A}_{N,k}^+$  is a subset of the dominant integrable positive weights of  $\mathfrak{gl}(N)$  and is identified with the space of the solution of  $q$ -boson Bethe Ansatz equation (2.22). Especially  $S_{0\nu}(t) = ||\psi_N(\{e^{2\pi i \varphi^a}\}, t)||^{-1}$ . The  $S_{\mu\nu}(t)$  can be regarded as one-parameter deformation of modular  $S$ -matrices, Because  $S_{\mu\nu}(t)$  reduces to modular  $S$ -matrices in  $t \rightarrow 0, (m \rightarrow \infty)$ . From the equivalence (2.20), the partition function of Chern-Simons-Matter theory is also expressed by deformed modular matrices as

$$\begin{aligned}\mathcal{Z}_{\text{CSM1}}(S^1 \times \Sigma_h) &= \sum_{\{\varphi^a\}_{a=1}^N \in \mathcal{B}_1} \langle \psi_N(\{e^{2\pi i \varphi^a}\}, t) | \psi_N(\{e^{2\pi i \varphi^a}\}, t) \rangle^{h-1} \\ &= \sum_{\mu \in \mathcal{A}_{N,k}^+} \frac{1}{S_{0\mu}^{2h-2}(t)}\end{aligned}\tag{2.25}$$

In the limit  $t \rightarrow 0$ , (2.25) reduces to (1.1).

We comment on the properties of partition function (2.25). Since  $S_{\mu\nu}(t)$  is function of the roots of Bethe Ansatz equation, it is difficult to compute the each deformed modular matrix  $S_{\mu\nu}(t)$  itself. But as we mention, (2.25) is explicitly calculable by using axiom of 2d TQFT. The 2d TQFT structure we concern is identical to a finite dimensional commutative Frobenius algebra constructed by Korff.<sup>1</sup> Remarkably we found that the partition function of  $U(N)/U(N)$  gauged WZW-matter model becomes a polynomial of  $t$  for  $h \geq 1$  with the integer coefficients and series with the integer coefficients for  $h = 0$ . The origin of integer valueness comes from properties of deformed fusion coefficient introduced in [15]:

$$N_{\mu\nu}^{\lambda}(t) := \sum_{\sigma \in \mathcal{A}_{N,k}^+} \frac{S_{\mu\sigma}(t) S_{\nu\sigma}(t) S_{\sigma\lambda}^{-1}(t)}{S_{0\sigma}(t)}. \quad (2.26)$$

$N_{\mu\nu}^{\lambda}(t)$  can be regarded as one-parameter deformation of Verline formula. The important point is that Korff derived the explicit algorithm to compute  $N_{\mu\nu}^{\lambda}(t)$  in terms of Bethe ansatz of  $q$ -boson and constructed the Frobenius algebra whose structure constant is given by  $N_{\mu\nu}^{\lambda}(t)$ .  $N_{\mu\nu}^{\lambda}(t)$  become polynomials with integer coefficients. In the TQFT side,  $N_{\mu\nu}^{\lambda}(t)$  corresponds to the data assigned to the pant (three punctured sphere). From the integer valueness of deformed fusion coefficients, we can show the integer valueness of the partition function of  $G/G$  gauged WZW-matter model.

We suggest a brief explanation of the integer valuedness of partition function in terms of Hilbert space interpretation of Chern-Simons-Matter theory. Recall that the partition function of pure Chern-Simons theory on  $S^1 \times \Sigma_h$  is expressed by trace over the Hilbert space on Riemann surface  $\mathcal{H}(\Sigma_h)$ .

$$\mathcal{Z}_{\text{CS}}(S^1 \times \Sigma_h) = \text{Tr}_{\mathcal{H}_{\text{CS}}(\Sigma_h)} 1. \quad (2.27)$$

It is known that the Hilbert space  $\mathcal{H}(\Sigma_h)$  is finite dimensional and same as the space of conformal block of WZW model on Riemann surface. Thus the partition agrees with the number of conformal blocks and integer. On the other hand, the partition function of twisted Chern-Simons-Matter theory is interpreted as the following index.

$$\mathcal{Z}_{\text{CSM1}}(S^1 \times \Sigma_h) = \text{Tr}_{\mathcal{H}_{\text{CSM}}(\Sigma_h)} e^{-2\pi Q_{U(1)} m}. \quad (2.28)$$

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<sup>1</sup>There is one to one correspondence between two dimensional topological quantum field theory and finite dimensional commutative Frobenius algebra

Here  $Q_{U(1)}$  is the charge associated to  $U(1)$ -rotation like this  $(\bar{\phi}, \phi) \rightarrow (e^{-i\alpha}\bar{\phi}, e^{i\alpha}\phi)$ . So each coefficient of the partition function should be an integer. But still integer valueness of partition function is rather mysterious in the path integral formalism. We will explain its geometric origin in terms of index over moduli space in the section 3.

## 2.2 The model 2

Next we consider a deformation of Chern-Simons theory which is related to a generalization of  $G/G$  gauged WZW model introduced in [10]. The matter fields are

- The Grassmann even matter:  $(\phi_{\bar{z}}, \phi_z, \bar{Y}, Y)$
- The Grassmann odd matter  $(\psi_{\bar{z}}, \psi_z, \bar{\chi}, \chi)$

All the matter fields again belong to adjoint representation of the Lie algebra. The BRST transformation is defined by

$$\begin{aligned} Q\phi_{\bar{z}} &= \psi_{\bar{z}}, & Q\psi_{\bar{z}} &= D_t\phi_{\bar{z}} + m\phi_{\bar{z}}, \\ QY &= D_t\chi - m\chi, & Q\chi &= Y, \\ Q\phi_z &= \psi_z, & Q\psi_z &= D_t\phi_z - m\phi_z, \\ Q\bar{Y} &= D_t\bar{\chi} + m\bar{\chi}, & Q\bar{\chi} &= \bar{Y}. \end{aligned} \quad (2.29)$$

Again we introduce the  $Q$ -exact matter action.

$$\begin{aligned} S_{\text{mat2}} &= \int_{S^1 \times \Sigma_h} Q\text{Tr} \left[ g^{z\bar{z}} \phi_z \psi_{\bar{z}} + \bar{Y} \chi \right] \\ &= \int_{S^1 \times \Sigma_h} \text{Tr} \left[ g^{z\bar{z}} \phi_z (D_t\phi_{\bar{z}} + m\phi_{\bar{z}}) + g^{z\bar{z}} \psi_z \psi_{\bar{z}} + \bar{Y} Y + \bar{\chi} (D_t\chi + m\chi) \right]. \end{aligned} \quad (2.30)$$

If we replace  $\phi_{\bar{z}}$  by  $\phi$ ,  $\psi_{\bar{z}}$  by  $\psi$  and so on, the action of model 2 becomes that of the model 1. The difference between two models are the spin of the fields.

The partition function is defined by

$$\mathcal{Z}_{\text{CSM2}}(S^1 \times \Sigma_h) = \frac{1}{\text{Vol}(\mathcal{G})} \int \mathcal{D}^3 A_\mu \mathcal{D}^2 \lambda_z \mathcal{D}^2 \phi_i \mathcal{D}^2 \psi_i \mathcal{D}^2 Y \mathcal{D}^2 \chi e^{S_{\text{coh}} + S_{\text{mat2}}} \quad (2.31)$$

The evaluation of the partition function is quite parallel to the first twist. Then the partition function is given by

$$\begin{aligned} \mathcal{Z}_{\text{CSM2}}(S^1 \times \Sigma_h) &= (1 - e^{-2\pi m})^{-N(h-1)} \\ &\times \sum_{\{\varphi^a\}_{a=1}^N \in \mathcal{B}_2} \left( \left| \det_{c,d} \left[ \frac{\partial \beta_2^c}{\partial \varphi^d} \right] \right| \prod_{a \neq b} \frac{1}{(e^{2\pi i \varphi_a} - e^{2\pi i \varphi_b})(e^{2\pi i \varphi_a} - e^{-2\pi m} e^{2\pi i \varphi_b})} \right)^{h-1} \end{aligned} \quad (2.32)$$

Here  $\{\varphi^a\}_{a=1}^N$  come from constant mode of gauge field  $A_t^a(z, \bar{z})$ . and  $\mathcal{B}_2$  is the set of the solutions of the following equations:

$$2\pi i(k + 2N)\varphi^a + \sum_{\substack{b=1 \\ b \neq a}}^N \log \frac{e^{2\pi i\varphi^a} - e^{2\pi i\varphi^b} e^{-2\pi m}}{e^{2\pi i\varphi^a} e^{-2\pi m} - e^{2\pi i\varphi^b}} = 2\pi i n^a, \quad n^a \in \mathbb{Z}. \quad (2.33)$$

## 2.3 The correspondence of Correlation functions

In this section we generalize the correspondence between partition function to correlation functions of  $Q$ -closed operator. The  $Q$ -closed gauge invariant operators in the Chern-Simons-matter theory are Wilson loops along the trivial fiber direction

$$W_\nu := \text{Tr}_\nu \text{P exp} \left( i \int_{S^1} A_t dt \right). \quad (2.34)$$

Here the trace  $\text{Tr}_\nu$  is taken over a representation  $\nu$  of gauge group. A normalized correlation function of Wilson loops on genus- $h$  Riemann surface is defined by

$$\begin{aligned} \langle \prod_{l=1}^n W_{\nu_l} \rangle &:= \frac{1}{\mathcal{Z}_{\text{CSM1}}(S^1 \times \Sigma_h) \text{Vol}(\mathcal{G})} \\ &\times \int \mathcal{D}^3 A_\mu \mathcal{D}^2 \lambda_i \mathcal{D}^2 \phi \mathcal{D}^2 \psi \mathcal{D}^2 Y_i \mathcal{D}^2 \chi_i e^{S_{\text{coh}} + S_{\text{mat}}} \prod_{l=1}^n W_{\nu_l} \end{aligned} \quad (2.35)$$

Again we take the gauge condition (2.11). Then the Wilson loop operator is written as

$$W_\nu = \text{Tr}_\nu \exp \left( 2\pi i \sum_{a=1}^N A_t^a(z, \bar{z}) H_a \right) \quad (2.36)$$

Since the path integral over the component fields can be performed same way as partition function case. The field configurations localized to equation  $\mathcal{B}_1$ . Then, the correlation function of Wilson loop in the model 1 is written as

$$\begin{aligned} \mathcal{Z}_{\text{CSM}}(\Sigma_h) \langle \prod_{l=1}^n W_{\nu_l} \rangle &= (1 - e^{-2\pi m})^{N(h-1)} \\ &\times \sum_{\{\varphi^a\}_{a=1}^N \in \mathcal{B}_1} \prod_{l=1}^n \text{Tr}_{\nu_l} e^{2\pi i \sum_a \phi^a} \left( \det_{c,d} \left[ \frac{\partial \beta_1^c(\varphi)}{\partial \varphi^d} \right] \prod_{a \neq b} \frac{e^{2\pi i\varphi_a} - e^{-2\pi m} e^{2\pi i\varphi_b}}{e^{2\pi i\varphi_a} - e^{2\pi i\varphi_b}} \right)^{h-1} \end{aligned} \quad (2.37)$$

As we mentioned before, the over all ambiguity does not affect the normalized correlation functions. The corresponding objects in the  $G/G$  gauged WZW-matter model is the correlation function of  $G$ -elements:

$$\langle \prod_{l=1}^n \text{Tr}_{\nu_l} g \rangle, \quad (g : \Sigma_h \rightarrow G). \quad (2.38)$$

Then The correlation function perfectly agree with

$$\langle \prod_{l=1}^n W_{\nu_l} \rangle = \langle \prod_{l=1}^n \text{Tr}_{\nu_l} g \rangle \quad (2.39)$$

We comment on the physical meaning of the above correlation in the  $q$ -boson model side. We showed that correlations function of  $G$ -elements is same as the correlation functions of conserved charges in the  $q$ -boson model. See [7] for the detailed identification . Thus we have established that Gauge/Bethe correspondence of correlation functions between a Chern-Simons-matter theory (model 1) and  $q$ -boson model.

### 3 Reation to moduli spaces

#### 3.1 integration over moduli pace

In this section we study the relation between moduli spaces defined by particular equations and the partition functions of Chern-Simons-Matter theories. Recall that the partition function of  $G/G$  gauged WZW model or equivalently pure Chern-Simons theory have a nice interpretation as a geometric index over the moduli space of flat connection on the Riemann surface and it reduces to the volume of flat connection BF theory limit in [6]. From this analogy, we conjectured that the partition function of  $G/G$  gauged WZW-matter model is related to index over moduli space on  $\Sigma_h$  of following equation in [7].

$$\mathcal{M}_1 = \left\{ (\bar{\phi}, \phi, A_i) \left| \epsilon^{ij} F_{ij} + \omega[\phi, \bar{\phi}] = 0, D_z \phi = 0, D_{\bar{z}} \bar{\phi} = 0 \right. \right\} / \mathcal{G} \quad (3.1)$$

Here  $\omega = \sqrt{\det(g_{\Sigma_h})}$  with the metric  $g_{\Sigma_h}$  on Riemann surface Note that  $(\bar{\phi}, \phi, A_i)$  in (3.1) are two dimensionanl fields on the Riemann surface and do not depend on the time direction. When the BF theory-like limit is taken, the  $G/G$  gauged WZW-matter model reduces to A-twisted gauged linear sigma model (GLSM) with an adjoint chiral multiplet. The partition function of the GLSM is interpreted as equivariant volume in the sense of [11]. we can show path integral of the GLSM localize to the moduli space  $\mathcal{M}_1$  by changing a  $Q$ -exact matter action. On the other hand, in level of the gauged WZW-matter model, it was unclear how the path integral of gauged WZW-matter model localized to  $\mathcal{M}_1$ . we will a give field theoretic explanation in terms of Chern-Simons-matter theories. In order to do this, we modify the previous  $Q$ -exact matter action as

$$S_{\text{matl}}^{\tau_0, \tau_1, \tau_2} = \int_{S^1 \times \Sigma_h} Q \text{Tr} \left[ \tau_0 \bar{\phi} \psi + \tau_1 g^{z\bar{z}} Y_z \chi_{\bar{z}} + \tau_2 (D_z \phi \cdot \chi_{\bar{z}} + D_{\bar{z}} \bar{\phi} \cdot \chi_z) \right] \quad (3.2)$$

with  $\tau_i \in \mathbb{C}$ . Since the action is  $Q$ -exact, the partition function is independent of the generic point in the parameter space  $(\tau_0, \tau_1, \tau_2)$  at least the action is bound from below. In the slice  $(\tau_0, \tau_1, \tau_2) = (1, 1, 0)$ , the action recover the original action (2.6). On the other hand, the matter action is written in the slice  $(\tau_0, \tau_1, \tau_2) = (\frac{k}{4\pi}, 0, i)$  as

$$S_{\text{mat1}}^{\tau_0=-\frac{k}{4\pi}, \tau_1=0, \tau_2=i} = \int_{S^1 \times \Sigma_h} \text{Tr} \left[ \frac{k}{4\pi} \bar{\phi} (D_t \phi + m \phi) + i g^{z\bar{z}} (D_z \phi Y_{\bar{z}} + D_{\bar{z}} \bar{\phi} Y_z) + \dots \right] \quad (3.3)$$

Here ellipses indicate the  $A_t, Y_z$  and  $Y_{\bar{z}}$  independent terms. Then the path integral over the  $A_t, Y_{\bar{z}}$  and  $Y_z$  impose the delta function constraints  $\delta(\omega[\phi, \bar{\phi}] + \epsilon^{ij} F_{ij})$ ,  $\delta(D_z \phi)$  and  $\delta(D_{\bar{z}} \bar{\phi})$ , respectively. The partition function is written as

$$\begin{aligned} \mathcal{Z}_{\text{CSM1}}(S^1 \times \Sigma_h) &= \int \mathcal{D}^2 A_i \mathcal{D}^2 \lambda_i \mathcal{D}^2 \phi \mathcal{D}^2 \psi \mathcal{D}^2 \chi_i \delta(\omega[\phi, \bar{\phi}] + \epsilon^{ij} F_{ij}) \delta(D_z \phi) \delta(D_{\bar{z}} \bar{\phi}) \\ &\quad \times \exp \left( \int_{S^1 \times \Sigma_h} d^3 x \text{Tr} \left[ \frac{ik}{4\pi} \epsilon^{ij} A_i \partial_t A_j - \frac{k}{4\pi} \sqrt{g(3)}, \bar{\phi} \partial_t \phi \dots \right] \right) \end{aligned} \quad (3.4)$$

Thus we find that the path integrals localize to the moduli space (3.1). After we perform the path integrals for the field configurations  $\prod_{z, \bar{z} \in \Sigma_h} d\Psi(t, z, \bar{z})$ .  $\mathcal{Z}_{\text{CSM1}}(S^1 \times \Sigma_h)$  is interpreted the partition function of effective quantum mechanics on  $S^1$  whose target space is  $\mathcal{M}_1$ . Similarly, in the model 2, we can show that the path integral localizes to the moduli space of Hitchin equation [19]:

$$\mathcal{M}_{\text{Hitchin}} = \left\{ (\phi_z, \phi_{\bar{z}}, A_i) \left| F_{z\bar{z}} + [\phi_z, \phi_{\bar{z}}] = 0, D_z \phi_{\bar{z}} = 0, D_{\bar{z}} \phi_z = 0 \right. \right\} / \mathcal{G} \quad (3.5)$$

In the pure Chern-Simons theory case, quantization of Chern-Simons theory corresponds to quantization of the moduli space of flat connection. We expect that the quantization of twisted Chern-Simons-matter theory is related to quantization of  $\mathcal{M}_1$  or  $\mathcal{M}_{\text{Hitchin}}$ . Here  $A_i \partial_t A_j$  and  $\bar{\phi} \partial_t \phi$  behaves as the kinetic term  $p_i \dot{q}_j$  in the Hamiltonian formalism of effective quantum mechanics. It is interesting to study quantization of moduli spaces and its relation to  $q$ -boson model. This problem is left to our future work. Again, the partition function of model 2 define the effective quantum mechanics whose target space is the moduli space of Hitchin equation on Riemann surface:

### 3.2 index over moduli space: $SU(2)$ case

We conjectured in [7] that the partition function of  $G/G$  gauged WZW-matter model is related to a geometric index [9]. In this subsection, we perform the identification of the partition function and index for  $G = SU(2)$  case.

The moduli space  $\mathfrak{M}$  relevant to  $G/G$  gauged WZW-matter model is roughly speaking the isomorphism classes of holomorphic  $G_{\mathbb{C}}$ -bundle on genus  $h$  Riemann surface and the bundle on  $\mathfrak{M}$  is

$$\lambda_t(T\mathfrak{M}) \otimes \mathcal{O}(k) \otimes E_x^*U \quad (3.6)$$

Here  $\lambda_t(T\mathfrak{M})$  is defined by  $K^0$ -classes  $[\wedge^n T\mathfrak{M}]$  of anti-symmetric products of the tangent bundle  $\wedge^n T\mathfrak{M}$  as:

$$\lambda_t(T\mathfrak{M}) := 1 + \sum_{n=1}^{\infty} (-t)^n [\wedge^n T\mathfrak{M}] \in K^0(T\mathfrak{M}; \mathbb{Q})[t], \quad (3.7)$$

$\mathcal{O}(k)$  is the line bundle with chern class  $k$ .

For simplicity, we consider the  $G = SU(2)$  case, The general cases will be studied in our future work. The index  $\text{Ind}(\mathfrak{M}, \lambda_t(T\mathfrak{M}) \otimes \mathcal{O}(k) \otimes E_x^*U)$  is given by

$$\begin{aligned} & \text{Ind}(\mathfrak{M}, \lambda_t(T\mathfrak{M}) \otimes \mathcal{O}(k) \otimes E_x^*U) \\ &= \sum_u (1-t)^{h-1} \left[ \frac{(1-tu^2)(1-tu^{-2})}{(u-u^{-1})^2} \right]^{h-1} \left[ 2k+4+4t \left( \frac{u^2}{1-tu^2} + \frac{u^{-2}}{1-tu^{-2}} \right) \right]^{h-1} \end{aligned} \quad (3.8)$$

Here  $\sum_u$  runs over the set of solution of following equation

$$u^{2k+4} \left[ \frac{1-tu^{-2}}{1-tu^2} \right]^2 = 1. \quad (3.9)$$

Next we calculate the partition function of  $SU(2)/SU(2)$  gauged WZW-matter model or equivalently the partition function of model 1, the ratio of functional determinants of gauge field and ghost field becomes

$$\prod_{\alpha \neq 0} \frac{\text{Det}_c(\partial_t + i\alpha(A_t))}{\text{Det}_{A_i}(\partial_t + i\alpha(A_t))} = \exp \left\{ \frac{1}{8\pi} \int_{\Sigma_h} R \log(1 - e^{4\pi i A_t}) (1 - e^{-4\pi i A_t}) + \frac{1}{2\pi} \int_{\Sigma_h} F \log \frac{1 - e^{4\pi i A_t}}{1 - e^{-4\pi i A_t}} \right\}. \quad (3.10)$$

The ratio of functional determinants of matter fields becomes

$$\begin{aligned} & \frac{\text{Det}_\chi(\partial_t + i\alpha(A_t) - m)}{\text{Det}_\phi(\partial_t + i\alpha(A_t) - m)} = (1 - e^{-2\pi m}) \\ & \times \exp \left\{ -\frac{1}{8\pi} \int_{\Sigma_h} R \log(1 - e^{4\pi i A_t - 2\pi m}) (1 - e^{-4\pi i A_t - 2\pi m}) - \frac{1}{2\pi} \int_{\Sigma_h} F \log \frac{1 - e^{4\pi i A_t - 2\pi m}}{1 - e^{-4\pi i A_t - 2\pi m}} \right\}. \end{aligned} \quad (3.11)$$



The  $\beta^a(\varphi)_1$  becomes

$$\begin{aligned}\beta^a(\varphi)_1 &= \frac{1}{2\pi} \left[ 2\pi i k \varphi + \log \left( \frac{1 - e^{4\pi i \varphi}}{1 - e^{-4\pi i \varphi}} \right) + \log \left( \frac{1 - e^{4\pi i \varphi - 2\pi m}}{1 - e^{-4\pi i \varphi - 2\pi m}} \right) \right] \\ &= \frac{1}{2\pi} \left[ 2\pi i (k + 2) \varphi + \pi i + 2 \log \left( \frac{1 - e^{4\pi i \varphi - 2\pi m}}{1 - e^{-4\pi i \varphi - 2\pi m}} \right) \right]\end{aligned}\quad (3.12)$$

Then partial integration and using poisson resummation formula impose the following field configuration

$$2\pi i (k + 2) \varphi + 2\pi i \log \left( \frac{1 - e^{4\pi i \varphi - 2\pi m}}{1 - e^{-4\pi i \varphi - 2\pi m}} \right) \in 2\pi i (n + \frac{1}{2}), \quad n \in \mathbb{Z} \quad (3.13)$$

and the partion function of  $SU(2)/SU(2)$  gauged WZW-matter model is given by

$$\frac{\partial \beta_1(\varphi)}{\partial \varphi} = i \left[ k + 2 + 2e^{-2\pi m} \left( \frac{e^{-4\pi i \varphi - 2\pi m}}{1 - e^{-4\pi i \varphi - 2\pi m}} + \frac{e^{4\pi i \varphi - 2\pi m}}{1 - e^{4\pi i \varphi - 2\pi m}} \right) \right] \quad (3.14)$$

Then the partition function is given by

$$\begin{aligned}\mathcal{Z}_{\text{CSM}}^{\text{SU}(2)}(S^1 \times \Sigma_h) &= (1 - e^{-2\pi m})^{h-1} \sum_{x \in \mathcal{B}_1} \left( \frac{(1 - e^{-2\pi m} e^{-4\pi i \varphi})(1 - e^{-2\pi m} e^{4\pi i \varphi})}{(e^{2\pi i \varphi} - e^{-2\pi i \varphi})^2} \right)^{h-1} \\ &\quad \times \left[ 2k + 4 + 4e^{-2\pi m} \left( \frac{e^{-4\pi i \varphi - 2\pi m}}{1 - e^{-4\pi i \varphi - 2\pi m}} + \frac{e^{4\pi i \varphi - 2\pi m}}{1 - e^{4\pi i \varphi - 2\pi m}} \right) \right]^{h-1}\end{aligned}\quad (3.15)$$

When we define  $u := e^{2\pi i \varphi}$  and  $t := e^{-2\pi m}$ ,  $\mathcal{Z}_{\text{CSM}}^{\text{SU}(2)}(S^1 \times \Sigma_h)$  agrees with (3.9) up to a overall factor. Thus the partition function has the interpretation of index.

## 4 Summary

In this paper we evaluated the partition functions of twisted Chern-Simons-matter theory on  $S^1 \times \Sigma_h$  and showed that the partition function agrees with the partition functions of  $G/G$  gauged WZW-matter model on  $\Sigma_h$ . We revealed Chern-Simons-matter theory on  $S^1 \times \Sigma_h$  also possess hidden quantum integrable structure. We also evaluated correlations of Wilson loops in twisted Chern-Simons-matter theory and showed that these correlation functions perfectly agree with the correlation functions of  $G$ -elements in a generalizations of  $G/G$  gauged WZW-model.

The partition function of the model 1 is also expressed by norms of eigen states of  $q$ -boson model and have TQFT structure. More over we explicitly showed that the partition function is can be interpreted as an index over moduli space of  $G$ -bundles for  $SU(2)$  case.

On the other hand, the physical interpretation of the partition function of model 2 is not clear in the quantum integrable side. It is interesting to find corresponding object of the partition function of model 2 in the quantum integrable model side. us

## Note added

When this work was being completed, there appeared a paper [20] which has substantial overlap with ours.

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